AMORPHIC ASSOCIATION SCHEMES WITH NEGATIVE LATIN SQUARE TYPE GRAPHS

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Dedicated to Zhe-Xian Wan on the occasion of his 80th birthday

ABSTRACT. Applying results from partial difference sets, quadratic forms, and recent results of Brouwer and Van Dam, we construct the first known amorphic association scheme with negative Latin square type graphs and whose underlying set is a nonelementary abelian 2-group. We give a simple proof of a result of Hamilton that generalizes Brouwer's result. We use multiple distinct quadratic forms to construct amorphic association schemes with a large number of classes.

1. INTRODUCTION

Let X be a finite set. A (*symmetric*) *association scheme* with d classes on X is a partition of $X \times X$ into sets $R_0, R_1, \ldots, R_d$ (called relations, or associate classes) such that

(1) $R_0 = \{(x,x) \mid x \in X\}$ (the diagonal relation);
(2) R_ℓ is symmetric for $\ell = 1, 2, \ldots, d$;
(3) for all i, j, k in $\{0, 1, 2, \ldots, d\}$ there is an integer p_{ij}^k such that, for all $(x,y) \in R_k$,

$$|\{z \in X \mid (x,z) \in R_i \text{ and } (z,y) \in R_j\}| = p_{ij}^k.$$

Since each symmetric relation R_ℓ, $1 \le \ell \le d$, corresponds to an (undirected) graph $G_\ell = (X, R_\ell)$, $1 \le \ell \le d$, with vertex set X and edge set R_ℓ, we can think of an association scheme $(X, \{R_\ell\}_{0 \le \ell \le d})$ as an edge-decomposition of the complete graph on the vertex set X into graphs G_ℓ on the same vertex set with the property that for all i, j, k in $\{1, 2, \ldots, d\}$ and for all $xy \in E(G_k)$,

$$|\{z \in X \mid xz \in E(G_i) \text{ and } zy \in E(G_j)\}| = p_{ij}^k,$$

where $E(G_k)$, $E(G_i)$, and $E(G_j)$ are the edge sets of G_k, G_i and G_j respectively. The graphs G_ℓ, $1 \le \ell \le d$, will be called *the graphs* of the association scheme $(X, \{R_\ell\}_{0 \le \ell \le d})$. (Note that it follows from the definition of an association scheme that each graph G_ℓ of the association scheme $(X, \{R_\ell\}_{0 \le \ell \le d})$ is regular with valency $n_\ell = p_{\ell\ell}^0$.) A *strongly regular graph* (SRG) is an association scheme with two classes, say $(X, \{R_\ell\}_{0 \le \ell \le 2})$. The elements of X are the vertices of the graph and $\{x, y\}$ is an edge if $(x,y) \in R_1$. There are four parameters (v, k, λ, μ) associated with the SRG, where

$$v = |X|, \ k = n_1, \ \lambda = p_{11}^1, \text{ and } \mu = p_{11}^2.$$

For more background on association schemes and strongly regular graphs, see [2] and [14]. Given an association scheme $(X, \{R_\ell\}_{0 \le \ell \le d})$, we can take the union of classes to form graphs with larger edge sets (this is called a *fusion*), but it is not necessarily guaranteed that the fused collection of graphs will form an association scheme on X. If an association scheme has the property that any of its fusions is also an association scheme, then we call the association scheme *amorphic*.

Key words and phrases. amorphic association scheme, association scheme, Latin square type partial difference set, negative Latin square type partial difference set, partial difference set, quadratic form, quadric, strongly regular graph.

A typical example of amorphic association schemes is given by the so-called uniform cyclotomy of finite fields.

Example 1.1. *Let $q = p^s$ be a power of a prime p and let $q-1 = ef$ with $e > 1$. Let $\mathbb{F}_q^*$ denote the multiplicative group of $\mathbb{F}_q$, C_0 be the subgroup of $\mathbb{F}_q^*$ of index e, and let $C_0, C_1, \ldots, C_{e-1}$ be the cosets of C_0 in $\mathbb{F}_q^*$. Assume $-1 \in C_0$. Define $R_0 = \{(x,x) \mid x \in \mathbb{F}_q\}$, and for $i \in \{1, 2, \ldots, e\}$, define $R_i = \{(x,y) \mid x, y \in \mathbb{F}_q, x - y \in C_{i-1}\}$. Then $(\mathbb{F}_q, \{R_i\}_{0 \le i \le e})$ is an association scheme with e classes. It was proved in* [1] *that, for $e > 2$, this scheme is amorphic if and only if -1 is a power of p modulo e.*

A (v, k, λ, μ) strongly regular graph is said to be of *Latin square type* (resp. *negative Latin square type*) if $(v, k, \lambda, \mu) = (n^2, r(n-\epsilon), \epsilon n + r^2 - 3\epsilon r, r^2 - \epsilon r)$ and $\epsilon = 1$ (resp. $\epsilon = -1$). If an association scheme is amorphic, then each of its graphs is clearly strongly regular. Moreover, A. V. Ivanov [11] showed that in an amorphic association scheme with at least three classes all graphs of the scheme are of Latin square type, or all graphs are of negative Latin square type. The converse of Ivanov's result is proved to be true in [10]. In fact even more is true because Van Dam [6] could prove the following result.

Theorem 1.2. *Let X be a set of size v, let $\{G_1, G_2, \ldots, G_d\}$ be an edge-decomposition of the complete graph on X, where each G_i is a strongly regular graph on X. If $G_i, 1 \le i \le d$, are all of Latin square type or all of negative Latin square type, then the decomposition is a d-class amorphic association scheme on X.*

We will consider the following situation. Let G be a finite additive group with identity 0. If we can partition $G \setminus \{0\}$ into sets D_i, all of which are Latin square type partial difference sets (defined below), or all of which are negative Latin square type partial difference sets, then the corresponding strongly regular Cayley graphs will satisfy the conditions of Theorem 1.2 and hence will form an amorphic association scheme. That is our objective in the constructions to follow.

1.1. **Partial Difference Sets.** A k-element subset D of a finite multiplicative group G of order v is called a (v, k, λ, μ)-*partial difference set* (PDS) in G provided that the list of "differences", $d_1 d_2^{-1}$, $d_1, d_2 \in D$, $d_1 \neq d_2$, contains each nonidentity element of D exactly λ times and each nonidentity element in $G \backslash D$ exactly μ times. Partial difference sets are equivalent to strongly regular graphs with a regular automorphism group via the Cayley graph construction. We refer the reader to Ma [15] for a survey of results on PDS. If a PDS has parameters $(n^2, r(n-\epsilon), \epsilon n + r^2 - 3\epsilon r, r^2 - \epsilon r)$ with $\epsilon = 1$ (resp. $\epsilon = -1$), then the PDS is said to be of *Latin square type* (resp. *negative Latin square type*). Latin square type PDSs have been constructed in a variety of nonisomorphic groups of order n^2 by taking the disjoint union of r subgroups of order n that pairwise intersect trivially ([15]). In contrast, most known constructions of negative Latin square type PDSs occur in elementary abelian p-groups. As far as we know, the first infinite family of negative Latin square type PDSs in nonelementary abelian p-groups was constructed recently in [7].

A complex character of an abelian group is a homomorphism from the group to the multiplicative group of complex roots of unity. The *principal character* is the character mapping every element of the group to 1. All other characters are called *nonprincipal.* Starting with the important work of Turyn [17], character sums have been a powerful tool in the study of difference sets of all types. The following lemma states how character sums can be used to verify that a subset of an abelian group is a PDS.

Lemma 1.3. *Let G be an abelian group of order v and D be a k-subset of G such that $\{d^{-1} \mid d \in D\} = D$ and $1 \notin D$. Then D is a (v, k, λ, μ)-PDS in G if and only if, for any complex*

character χ of G,

$$\sum_{d\in D}\chi(d)=\begin{cases} k, & \text{if } \chi \text{ is principal on } G,\\ \frac{(\lambda-\mu)\pm\sqrt{(\lambda-\mu)^2+4(k-\mu)}}{2}, & \text{if } \chi \text{ is nonprincipal on } G.\end{cases}$$

In particular, let $v=n^2$ and let $k=r(n-\epsilon)$, where $\epsilon=\pm1$. Then D is a $(n^2, r(n-\epsilon), \epsilon n+r^2-3\epsilon r, r^2-\epsilon r)$-PDS in G if and only if, for any complex character χ of G,

$$\sum_{d\in D}\chi(d)=\begin{cases} r(n-\epsilon), & \text{if } \chi \text{ is principal on } G,\\ \epsilon(n-r) \text{ or } -\epsilon r, & \text{if } \chi \text{ is nonprincipal on } G.\end{cases}$$

We will use Lemma 1.3 frequently in the rest of the paper.

1.2. **Galois Ring Preliminaries.** Interested readers are referred to [16] for more details on Galois rings. We will follow the approach outlined in [7], including the fact that we will only use Galois rings over $\mathbb{Z}_4$. A *Galois ring over* $\mathbb{Z}_4$ *of degree* t, $t\geq 2$, denoted $\mathrm{GR}(4,t)$, is the quotient ring $\mathbb{Z}_4[x]/\langle\Phi(x)\rangle$, where $\Phi(x)$ is a basic primitive polynomial in $\mathbb{Z}_4[x]$ of degree t. Let ξ be a root of $\Phi(x)$ in $\mathrm{GR}(4,t)$. Then we have $\mathrm{GR}(4,t)=\mathbb{Z}_4[\xi]$. In this paper, we will only need $\mathrm{GR}(4,2)=\mathbb{Z}_4[x]/\langle x^2+x+1\rangle$.

The ring $R=\mathrm{GR}(4,2)$ is a finite local ring with unique maximal ideal $2R$, and $R/2R$ is isomorphic to the finite field $\mathbb{F}_4$. If we denote the natural epimorphism from R to $R/2R\cong\mathbb{F}_4$ by π, then $\alpha=\pi(\xi)$ is a primitive element of $\mathbb{F}_4$.

The set $\mathcal{T}=\{0,1,\xi,\xi^2\}$ is a complete set of coset representatives of $2R$ in R. This set is usually called a *Teichmüller system* of R. The restriction of π to $\mathcal{T}$ is a bijection from $\mathcal{T}$ to $\mathbb{F}_4$, and we refer to this bijection as $\pi_{\mathcal{T}}$. An arbitrary element β of R has a unique 2-adic representation

$$\beta=\beta_1+2\beta_2,$$

where $\beta_1,\beta_2\in\mathcal{T}$. Combining $\pi_{\mathcal{T}}$ with this 2-adic representation, we get a bijection F from $\mathbb{F}_4^{2\ell}$ to $R\times\mathbb{F}_4^{2\ell-2}$ defined by

$$F:(x_1,x_2,\ldots,x_{2\ell})\mapsto \quad (\pi_{\mathcal{T}}^{-1}(x_1)+2\pi_{\mathcal{T}}^{-1}(x_2),x_3,x_4,\ldots,x_{2\ell}).$$

The inverse of this map is the map F^{-1} from $R\times\mathbb{F}_4^{2\ell-2}$ to $\mathbb{F}_4^{2\ell}$,

$$F^{-1}:(\xi_1+2\xi_2,x_3,\ldots,x_{2\ell})\mapsto \quad (\pi_{\mathcal{T}}(\xi_1),\pi_{\mathcal{T}}(\xi_2),x_3,\ldots,x_{2\ell}).$$

To simplify notation we will usually omit the subindex in the bijection $\pi_{\mathcal{T}}^{-1}:\mathbb{F}_4\to\mathcal{T}$. (So from now on, π^{-1} means the inverse of the bijection $\pi_{\mathcal{T}}:\mathcal{T}\to\mathbb{F}_4$.) We will show in Section 3 how we can use F as a character-sum-preserving bijection to construct a PDS in a nonelementary abelian group, namely the additive group of $R\times\mathbb{F}_4^{2\ell-2}$.

The *Frobenius map* f from R to itself is the ring automorphism $f:\beta_1+2\beta_2\mapsto\beta_1^2+2\beta_2^2$. This map is used to define the *trace* Tr from R to $\mathbb{Z}_4$, namely, $\mathrm{Tr}(\beta)=\beta+\beta^f$, for $\beta\in R$. We note that here the Galois ring trace $\mathrm{Tr}:R\to\mathbb{Z}_4$ is related to the finite field trace $\mathrm{tr}:\mathbb{F}_4\to\mathbb{F}_2$ via

$$\mathrm{tr}\circ\pi=\pi\circ\mathrm{Tr}\,. \tag{1.1}$$

As a consequence, we have

$$\sqrt{-1}^{\,\mathrm{Tr}(2x)}=\sqrt{-1}^{\,2\,\mathrm{Tr}(x)}=(-1)^{\mathrm{Tr}(x)}=(-1)^{\pi\circ\mathrm{Tr}(x)}=(-1)^{\mathrm{tr}(\pi(x))},$$

for all $x\in R$.

All additive characters of Galois rings and finite fields can be defined by using the appropriate trace, as indicated in the following well-known lemma.

Lemma 1.4. (1) *Let χ be an additive character of $\mathbb{F}_q$, where q is a power of p, p is a prime, and let η_p be a complex primitive p^{th} root of unity. Then there is a $w \in \mathbb{F}_q$ such that $\chi(x) = \eta_p^{\operatorname{tr}(wx)}$ for all $x \in \mathbb{F}_q$, where tr is the trace from $\mathbb{F}_q$ to $\mathbb{F}_p$.*

(2) *Let ψ be an additive character of $\mathrm{GR}(4,t)$. Then there is a $\beta \in \mathrm{GR}(4,t)$ such that $\psi(x) = \sqrt{-1}^{\operatorname{Tr}(\beta x)}$ for all $x \in \mathrm{GR}(4,t)$.*

Since we can write $\beta = \beta_1 + 2\beta_2$ for any $\beta \in R = \mathrm{GR}(4,2)$, where $\beta_k \in \mathcal{T}, k = 1,2$, we will use the notation $\psi_\beta = \psi_{\beta_1+2\beta_2}$ indicating the ring element used to define the character $x \mapsto \sqrt{-1}^{\operatorname{Tr}(\beta x)}$. If $\beta_1 = 0$ but $\beta_2 \neq 0$, then $\psi_{2\beta_2}$ is a character of order 2 and $\psi_{2\beta_2}$ is principal on $2R$. If $\beta_1 \neq 0$, then $\psi_{\beta_1+2\beta_2}$ is a character of order 4 and $\psi_{\beta_1+2\beta_2}$ is nonprincipal on $2R$. For the finite field $\mathbb{F}_4$, we will use the notation χ_w to indicate the field element used to define the character $x \mapsto (-1)^{\operatorname{tr}(wx)}$. Characters of $R \times \mathbb{F}_4^{2\ell-2}$ will be written as

$$\Psi = \psi_{\beta_1+2\beta_2} \otimes \chi_{w_3} \otimes \chi_{w_4} \otimes \cdots \otimes \chi_{w_{2\ell}} = \psi_{\beta_1+2\beta_2} \otimes \chi_{(w_3,w_4,\ldots,w_{2\ell})}, \tag{1.2}$$

where $\beta_i \in \mathcal{T}, 1 \leq i \leq 2$, and $w_i \in \mathbb{F}_4, 3 \leq i \leq 2\ell$.

1.3. **Quadratic Forms.** Let $\mathbb{F}_q$ be the field of order q, where q is a prime power, and let V be an m-dimensional vector space over $\mathbb{F}_q$. A function $Q : V \to \mathbb{F}_q$ is called a *quadratic form* if

(1) $Q(\gamma v) = \gamma^2 Q(v)$ for all $\gamma \in \mathbb{F}_q$ and $v \in V$,
(2) the function $B : V \times V \to \mathbb{F}_q$ defined by $B(v_1, v_2) = Q(v_1 + v_2) - Q(v_1) - Q(v_2)$ is bilinear.

We call Q *nonsingular* if the subspace W with the property that Q vanishes on W and $B(w, v) = 0$ for all $v \in V$ and $w \in W$ is the zero subspace. For background on quadratic forms, see [5]. Let Q be a nonsingular quadratic form on an m-dimensional vector space V over $\mathbb{F}_q$. If m is even, then Q is equivalent to either $x_1x_2 + x_3x_4 + \cdots + x_{m-1}x_m$ (called *hyperbolic*, or type $+1$) or $p(x_1, x_2) + x_3x_4 + \cdots + x_{m-3}x_{m-2} + x_{m-1}x_m$, where $p(x_1, x_2)$ is an irreducible quadratic form in two indeterminates (called *elliptic*, or type -1).

Let $\mathrm{PG}(m-1, q)$ be the $(m-1)$-dimensional desarguesian projective space over $\mathbb{F}_q$, and let V be its associated vector space. The *quadric* of the projective space $\mathrm{PG}(m-1, q)$ defined by a quadratic form $Q : V \to \mathbb{F}_q$ is the point set $\mathcal{Q} = \{\langle v\rangle \in \mathrm{PG}(m-1, q) \mid Q(v) = 0\}$. The following theorem about quadrics in $\mathrm{PG}(m-1, q)$ is well known, see for example [4].

Theorem 1.5. *Let $\mathcal{Q}$ be a nonsingular elliptic quadric in $\mathrm{PG}(2\ell-1, q)$, and let Ω denote the set of nonzero vectors in $\mathbb{F}_q^{2\ell}$ corresponding to $\mathcal{Q}$ (i.e., $\Omega = \{x \in \mathbb{F}_q^{2\ell} \mid x \neq 0, \langle x\rangle \in \mathcal{Q}\}$). For any nontrivial additive character χ of $\mathbb{F}_q^{2\ell}$, we have*

$$\chi(\Omega) = \begin{cases} (q^{\ell-1} - 1) - q^\ell, & \text{or} \\ (q^{\ell-1} - 1). \end{cases}$$

That is, Ω is a $(q^{2\ell}, (q^\ell+1)(q^{\ell-1}-1), q^{2\ell-2} - q^{\ell-1}(q-1) - 2, q^{2\ell-2} - q^{\ell-1})$-negative Latin square type PDS in the additive group of $\mathbb{F}_q^{2\ell}$.

There is a corresponding theorem for hyperbolic quadratic forms, where the set Ω in that case will be a Latin square type PDS. Since the main focus of this paper is on constructions of negative Latin square type PDSs, we do not state that theorem nor do we include the constructions of Latin square type amorphic association schemes.

Brouwer [3] showed that one can also use quadratic forms to define other PDSs than the ones listed in the previous theorem. Suppose $\mathbb{F}_{q_0}$ is a subfield of $\mathbb{F}_q$, say $q = q_0^e$. Let $V = \mathbb{F}_q^{2\ell}$, $Q : V \to \mathbb{F}_q$ be a nonsingular quadratic form on V, and $\operatorname{tr}_{q/q_0}$ be the trace from $\mathbb{F}_q$ to $\mathbb{F}_{q_0}$.

Then $Q_0 = \mathrm{tr}_{q/q_0} \circ Q : V \to \mathbb{F}_{q_0}$ is a quadratic form on the same V but now viewed as a $2\ell e$-dimensional vector space over $\mathbb{F}_{q_0}$. Define

$$\Omega = \{x \in V \mid Q(x) = 0, x \neq 0\},\ \Omega_0 = \{x \in V \mid Q_0(x) = 0, x \neq 0\}.$$

Then clearly $\Omega \subset \Omega_0$. Brouwer [3] proved the following theorem.

Theorem 1.6. *With the assumptions and notation above, the set $\Omega_0 \backslash \Omega$ is a PDS in the additive group of V.*

Remark 1.7. *In the above theorem, Q can be either elliptic or hyperbolic. If Q is a nonsingular elliptic quadratic form on $V = \mathbb{F}_q^{2\ell}$, then*

$$|\Omega_0 \setminus \Omega| = (q_0^{\ell e} + 1)(q_0^{\ell e-1} - q_0^{\ell e - e}),$$

and $\Omega_0 \setminus \Omega$ is a negative Latin square type PDS in $(V, +)$. This will be the case to which we pay most attention. If Q is a nonsingular hyperbolic quadratic form on $V = \mathbb{F}_q^{2\ell}$, then

$$|\Omega_0 \setminus \Omega| = (q_0^{\ell e} - 1)(q_0^{\ell e-1} - q_0^{\ell e - e}),$$

and $\Omega_0 \setminus \Omega$ is a Latin square type PDS in $(V, +)$.

We will see in the following sections how this result leads to amorphic association schemes. In Section 2, we provide a relatively simple example that motivated the work done in this paper. Section 3 uses techniques similar to that in [7] to construct an amorphic association scheme whose underlying set is a nonelementary abelian group and whose strongly regular graphs have negative Latin square parameters. To our knowledge, no such example was previously known. Section 4 provides examples of constructions in larger fields, and it also includes a different proof of a theorem shown in [9].

2. A 4-class Amorphic Association Scheme

In this section we use Theorem 1.6 to construct a 4-class amorphic association scheme in the additive group of $V = \mathbb{F}_4^{2\ell}$, where $\ell \geq 2$. According to Theorem 1.2, we can do this by partitioning $V \setminus \{0\}$ into 4 partial difference sets with negative Latin square type parameters. Let $\mathbb{F}_4 = \{0, 1, \alpha, \alpha^2 = \alpha + 1\}$, and let

$$Q(x_1, x_2, \ldots, x_{2\ell}) = \alpha x_1^2 + x_1 x_2 + x_2^2 + x_3 x_4 + \cdots + x_{2\ell-1} x_{2\ell}.$$

This is an elliptic quadratic form on V. By Theorem 1.5, the set $\Omega = \{x \in V \mid Q(x) = 0, x \neq 0\}$ is a negative Latin square type PDS. We introduce the notation

$$D_\beta = \{v \in \mathbb{F}_4^{2\ell} \mid Q(v) = \beta\},$$

where $\beta \in \mathbb{F}_4$. Note that $\Omega = D_0 \backslash \{0\}$ and $D_1 = \{v \in \mathbb{F}_4^{2\ell} \mid \mathrm{tr}_{4/2} \circ Q(v) = 0\} \setminus D_0$. Theorem 1.6 shows that the set D_1 is a negative Latin square type PDS. Now we would like to show that $D_\alpha = \{x \in V \mid Q(x) = \alpha\}$ and $D_{\alpha^2} = \{x \in V \mid Q(x) = \alpha^2\}$ are each negative Latin square type PDSs. We can do this by using the quadratic forms $Q' = \alpha Q$ and $Q'' = \alpha^2 Q$ and applying Theorems 1.5 and 1.6 to these quadratic forms in the same way we used Q to show that D_1 is a negative Latin square type PDS. Combining these PDSs with Theorem 1.2, we have proved the following theorem.

Theorem 2.1. *The strongly regular Cayley graphs generated by the negative Latin square type PDSs $D_0 \setminus \{0\}$, D_1, D_α, and D_{α^2} form a 4-class amorphic association scheme on V.*

Remark 2.2. *The subset $D_0 \setminus \{0\}$ has size $(4^\ell + 1)(4^{\ell-1} - 1)$, while the subsets D_1, D_α, and D_{α^2} of V all have the same size, namely $(4^\ell + 1)4^{\ell-1}$.*

3. Amorphic association schemes defined on nonelementary abelian 2-groups

In this section, we demonstrate that we can have an amorphic association scheme on $(G,+)$, in which all graphs are negative Latin Square type SRGs, and G is a nonelementary abelian 2-group. The construction in this section follows the techniques found in [7], where the set $D_0 \setminus \{0\}$ from the previous section was "lifted" from a PDS in $(\mathbb{F}_4^{2\ell},+)$ to a PDS in the group $\mathbb{Z}_4^2 \times \mathbb{Z}_2^{4\ell-4}$ by using the map F defined in Section 1.2. We will only include the smallest case where $G = \mathbb{Z}_4^2 \times \mathbb{Z}_2^{4\ell-4}$ since the general case (i.e., $G = \mathbb{Z}_4^{2k} \times \mathbb{Z}_2^{4\ell-4k}$, $1 < k < \ell$) can be handled in a way completely analogous to that in [7].

Again let $\ell \geq 2$ be an integer, $\mathbb{F}_4 = \{0, 1, \alpha, \alpha^2 = \alpha + 1\}$,

$$Q(x_1, x_2, \ldots, x_{2\ell}) = \alpha x_1^2 + x_1x_2 + x_2^2 + x_3x_4 + \cdots + x_{2\ell-1}x_{2\ell},$$

and let $D_\beta = \{v \in \mathbb{F}_4^{2\ell} \mid Q(v) = \beta\}$, where $\beta \in \mathbb{F}_4$. We define

$$\mathcal{L}_\beta = F(D_\beta)$$

for every $\beta \in \mathbb{F}_4$. For example,

$$\mathcal{L}_1 = \{(\xi_1 + 2\xi_2, x_3, \ldots, x_{2\ell}) \in R \times \mathbb{F}_4^{2\ell-2} \mid Q(\pi(\xi_1), \pi(\xi_2), x_3, \ldots, x_{2\ell}) = 1\},$$

where $R = \mathrm{GR}(4,2)$. The goal of this section is to demonstrate that $\mathcal{L}_\beta$, $\beta \in \{1, \alpha, \alpha^2\}$, are all negative Latin square type PDSs. We will provide the proof that $\mathcal{L}_1$ is a PDS in the additive group of $R \times \mathbb{F}_4^{2\ell-2}$ in its full details. It can be shown that $\mathcal{L}_\alpha$ and $\mathcal{L}_{\alpha^2}$ are also PDS by the same arguments with Q replaced by αQ and $\alpha^2 Q$ respectively. The careful reader will notice that an important part of the proofs (i.e., Lemma 3.3) we are going to give is different from that in [7]. When dealing with the "lifting" of $D_0 \setminus \{0\}$, the geometry of quadrics can help us with the proof (see Lemma 2.7 of [7]). It seems difficult to find similar geometric arguments which also work for $\mathcal{L}_1$. We have to rely on algebraic computations.

By Lemma 1.3, in order to demonstrate that $\mathcal{L}_1$ is a PDS we need to show that $\Psi(\mathcal{L}_1)$ are as required for all nonprincipal characters Ψ of $R \times \mathbb{F}_4^{2\ell-2}$. If we write $\Psi = \psi_{\beta_1+2\beta_2} \otimes \chi_{w_3,w_4,\ldots,w_{2\ell}}$, the character sum of $\mathcal{L}_1$ is as follows.

$$\Psi(\mathcal{L}_1) = \Sigma_{Q(\pi(\xi_1),\pi(\xi_2),x_3,\ldots,x_{2\ell})=1} \sqrt{-1}^{\mathrm{Tr}(\beta_1\xi_1)} (-1)^{\mathrm{tr}(\pi(\beta_2)\pi(\xi_1)+\pi(\beta_1)\pi(\xi_2)+w_3x_3+\cdots+w_{2\ell}x_{2\ell})}$$

If $\beta_1 = 0$, that is, Ψ is a character of order 2, then $\Psi(\mathcal{L}_1) = \chi_{\pi(\beta_2),0,w_3,\ldots,w_{2\ell}}(D_1)$. Since we know that D_1 is a PDS by Theorem 1.6, we have $\Psi(\mathcal{L}_1) = 4^{\ell-1}$ or $4^{\ell-1} - 4^\ell$ as required (cf. Lemma 1.3). Thus, the characters of order 2 all have the correct sum over $\mathcal{L}_1$. So from now on we assume that $\beta_1 \neq 0$. Then $\Psi(\mathcal{L}_1)$ can be broken down into three sub-sums according as $\mathrm{Tr}(\beta_1\xi_1)$ is $0, 2$, or odd.

$$\begin{aligned}
\Psi(\mathcal{L}_1) = & \sum_{(\xi_1+2\xi_2,x_3,\ldots,x_{2\ell})\in\mathcal{L}_1, \mathrm{Tr}(\beta_1\xi_1)=0} (-1)^{\mathrm{tr}(\pi(\beta_2\xi_1+\beta_1\xi_2))}(-1)^{\mathrm{tr}(\sum_{i=3}^{2\ell} w_ix_i)} \\
& - \sum_{(\xi_1+2\xi_2,x_3,\ldots,x_{2\ell})\in\mathcal{L}_1, \mathrm{Tr}(\beta_1\xi_1)=2} (-1)^{\mathrm{tr}(\pi(\beta_2\xi_1+\beta_1\xi_2))}(-1)^{\mathrm{tr}(\sum_{i=3}^{2\ell} w_ix_i)} \\
& + \sum_{(\xi_1+2\xi_2,\xi_3,\ldots,\xi_{2\ell})\in\mathcal{L}_1, \mathrm{Tr}(\beta_1\xi_1)=\mathrm{odd}} (\sqrt{-1})^{\mathrm{Tr}(\beta_1\xi_1)}(-1)^{\mathrm{tr}(\pi(\beta_2\xi_1+\beta_1\xi_2))}(-1)^{\mathrm{tr}(\sum_{i=3}^{2\ell} w_ix_i)}
\end{aligned}$$

The third sub-sum can be shown to be zero in exactly the same way as in [7]: namely, use the fact that $Q(x_1, x_1 + x_2, x_3, \ldots, x_{2\ell}) = Q(x_1, x_2, \ldots, x_{2\ell})$ to identify a pair of elements whose corresponding terms in the third sub-sum add to 0. For the first sub-sum, note that $\mathrm{Tr}(\beta_1\xi_1) = 0$ implies $\xi_1 = 0$. Define

$$O_0 = \{(x_1, x_2, \ldots, x_{2\ell}) \in D_1 \mid x_1 = 0\}.$$

Then the first sub-sum is equal to $\chi_{\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell}}(O_0)$. For the second sub-sum, note that $\mathrm{Tr}(\beta_1\xi_1)=2$ implies $\xi_1=\beta_1^{-1}$. We have

$$\begin{aligned}
\text{The second sub-sum} &= \sum_{(x_1,x_2,\ldots,x_{2\ell})\in D_1,\ x_1=\pi(\beta_1)^{-1}} \chi_{\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell}}((x_1,x_2,\ldots,x_{2\ell}))\\
&= \sum_{(x_1,x_2,\ldots,x_{2\ell})\in D_1,\ x_1\neq 0} \chi_{\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell}}((x_1,x_2,\ldots,x_{2\ell}))\\
&\quad - \sum_{(x_1,x_2,\ldots,x_{2\ell})\in D_1,\ x_1\neq 0,\ \mathrm{tr}(\pi(\beta_1)x_1)=1} \chi_{\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell}}((x_1,x_2,\ldots,x_{2\ell}))\\
&= \sum_{(x_1,x_2,\ldots,x_{2\ell})\in D_1,\ x_1\neq 0} \chi_{\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell}}((x_1,x_2,\ldots,x_{2\ell}))
\end{aligned}$$

The last equality follows since

$$\sum_{(x_1,x_2,\ldots,x_{2\ell})\in D_1,\ x_1\neq 0,\ \mathrm{tr}(\pi(\beta_1)x_1)=1} \chi_{\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell}}((x_1,x_2,\ldots,x_{2\ell}))=0$$

by the pairing argument used in dealing with the third sub-sum. Hence

$$\text{The second sub-sum} = \chi_{\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell}}(D_1\backslash O_0) \tag{3.1}$$

Thus,

$$\Psi(\mathcal{L}_1) = \chi_{\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell}}(O_0) - \chi_{\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell}}(D_1\backslash O_0) \tag{3.2}$$

Contrast this with

$$\chi_{\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell}}(D_1) = \chi_{\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell}}(O_0) + \chi_{\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell}}(D_1\backslash O_0) \tag{3.3}$$

Plugging $x_1=0$ into $Q(x_1,x_2,\ldots,x_{2\ell})=1$ yields $x_2^2+x_3x_4+\cdots+x_{2\ell-1}x_{2\ell}=1$, leading to the following simplification:

$$\chi_{\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell}}(O_0) = (-1)^{\mathrm{tr}(\pi(\beta_1))}\Sigma_{(x_3,\ldots,x_{2\ell})\in\mathbb{F}_4^{2\ell-2}}(-1)^{\mathrm{tr}(\pi(\beta_1)^2(x_3x_4+\cdots+x_{2\ell-1}x_{2\ell})+w_3^2x_3^2+\cdots+w_{2\ell}^2x_{2\ell}^2)}$$

Since $\pi(\beta_1)^2(x_3x_4+\cdots+x_{2\ell-1}x_{2\ell})+w_3^2x_3^2+\cdots+w_{2\ell}^2x_{2\ell}^2$ is a nonsingular quadratic form, Theorem 3.2 of [12] implies that

$$\chi_{\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell}}(O_0)=\pm 4^{\ell-1}$$

In the following two lemmas, we will find the values of $\chi_{\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell}}(D_1\setminus O_0)$ according as $\chi_{\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell}}(O_0)=-4^{\ell-1}$ or $4^{\ell-1}$.

Lemma 3.1. *If* $\chi_{\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell}}(O_0)=-4^{\ell-1}$, *then* $\chi_{\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell}}(D_1\backslash O_0)=\pm 2\cdot 4^{\ell-1}$.

Proof: Since D_1 is a PDS, the character sum of D_1 must be either $4^{\ell-1}$ or $4^{\ell-1}-4^{\ell}$ (cf. Lemma 1.3). Note that

$$\chi_{\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell}}(D_1\backslash O_0)=\chi_{\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell}}(D_1)-\chi_{\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell}}(O_0).$$

Therefore, if $\chi_{\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell}}(O_0)=-4^{\ell-1}$, then $\chi_{\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell}}(D_1\backslash O_0)=\pm 2\cdot 4^{\ell-1}$. □

Together with (3.2), the above lemma immediately leads to the following corollary.

Corollary 3.2. *If* $\chi_{\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell}}(O_0)=-4^{\ell-1}$, *then* $\Psi(\mathcal{L}_1)=4^{\ell-1}$ *or* $4^{\ell-1}-4^{\ell}$.

Next we consider the case where $\chi_{\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell}}(O_0)=4^{\ell-1}$. Again using (3.3), we see that in this case $\chi_{\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell}}(D_1\backslash O_0)=-4^{\ell}$ or 0. We will show that the case $\chi_{\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell}}(D_1\backslash O_0)=-4^{\ell}$ can not occur. The proof is somewhat lengthy.

Lemma 3.3. *If* $\chi_{\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell}}(O_0) = 4^{\ell-1}$*, then* $\chi_{\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell}}(D_1 \backslash O_0) = 0$.

Proof: From (3.1), we see that the character sum over $D_1 \backslash O_0$ is equal to the character sum over the elements of D_1 that satisfy $x_1 = \pi(\beta_1)^{-1}$. Plugging this condition into the equation $Q(x_1, x_2, \ldots, x_{2\ell}) = 1$ yields

$$\pi(\beta_1)x_2 + \pi(\beta_1)^2 x_2^2 + \alpha + \pi(\beta_1)^2 + \pi(\beta_1)^2(x_3x_4 + \cdots + x_{2\ell-1}x_{2\ell}) = 0 \tag{3.4}$$

For convenience, set

$$\gamma := \alpha + \pi(\beta_1)^2 + \pi(\beta_1)^2(x_3x_4 + \cdots + x_{2\ell-1}x_{2\ell})$$

Using (3.4) in the sum $\chi_{\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell}}(D_1 \backslash O_0)$, and using properties of the finite field trace, we have

$$\chi_{\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell}}(D_1 \backslash O_0) = (-1)^{\mathrm{tr}(\pi(\beta_2)\pi(\beta_1)^{-1})} \cdot 2 \cdot (S_1 - S_2), \tag{3.5}$$

where

$$S_1 = \Sigma_{(x_3,x_4,\ldots,x_{2\ell}) \in \mathbb{F}_4^{2\ell-2},\ \gamma=0} (-1)^{\mathrm{tr}(w_3x_3 + \cdots + w_{2\ell}x_{2\ell})},$$

and

$$S_2 = \Sigma_{(x_3,x_4,\ldots,x_{2\ell}) \in \mathbb{F}_4^{2\ell-2},\ \gamma=1} (-1)^{\mathrm{tr}(w_3x_3 + \cdots + w_{2\ell}x_{2\ell})}.$$

Following a similar computation over the elements $(0, x_2, \ldots, x_{2\ell}) \in O_0$ and defining γ as above (here $x_2^2 = 1 + x_3x_4 + \cdots + x_{2\ell-1}x_{2\ell}$ implies that $\gamma = \alpha + \pi(\beta_1)^2 x_2^2$), we see that

$$\begin{aligned} \chi_{\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell}}(O_0) &= \Sigma_{(x_3,x_4,\ldots,x_{2\ell}) \in \mathbb{F}_4^{2\ell-2},\ \gamma \in \{\alpha,\alpha^2\}} (-1)^{\mathrm{tr}(w_3x_3 + \cdots + w_{2\ell}x_{2\ell})} \\ &- \Sigma_{(x_3,x_4,\ldots,x_{2\ell}) \in \mathbb{F}_4^{2\ell-2},\ \gamma \in \{0,1\}} (-1)^{\mathrm{tr}(w_3x_3 + \cdots + w_{2\ell}x_{2\ell})} \end{aligned} \tag{3.6}$$

The main difference between the sum over $D_1 \setminus O_0$ and the sum over O_0 is that in the former case we know that γ is either 0 or 1 because it is equal to $\pi(\beta_1)^2 x_2^2 + \pi(\beta_1)x_2 = \mathrm{tr}(\pi(\beta_1)x_2)$ by (3.4); while in the latter case, γ can be any element in $\mathbb{F}_4$. For convenience, we will call the first sum in (3.6) S_3, and the second sum S_4. It follows from the definition of S_1, S_2, and S_4 that

$$S_1 + S_2 = S_4$$

By assumption, $\chi_{\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell}}(O_0) = 4^{\ell-1}$, we have

$$S_3 - S_4 = 4^{\ell-1}$$

If we add S_3 and S_4, we are simply taking the character sum over all of $\mathbb{F}_4^{2\ell-2}$. So

$$S_3 + S_4 = \begin{cases} 4^{2\ell-2}, & \text{if } (w_3, w_4, \ldots, w_{2\ell}) = (0, 0, \ldots, 0), \\ 0, & \text{otherwise.} \end{cases} \tag{3.7}$$

Hence we have

$$2(S_1 + S_2) = 2S_4 = \begin{cases} 4^{2\ell-2} - 4^{\ell-1}, & \text{if } (w_3, w_4, \ldots, w_{2\ell}) = (0, 0, \ldots, 0), \\ -4^{\ell-1}, & \text{otherwise.} \end{cases} \tag{3.8}$$

We now return to the computation of $\chi_{\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell}}(D_1 \backslash O_0)$. Define

$$H(x_3, x_4, \ldots, x_{2\ell}) = \pi(\beta_1)^2(x_3x_4 + \cdots + x_{2\ell-1}x_{2\ell}).$$

Note that H is a nonsingular hyperbolic quadratic form on $\mathbb{F}_4^{2\ell-2}$, and

$$\begin{aligned} S_1 &= \sum_{H(x_3,\ldots,x_{2\ell}) = \alpha + \pi(\beta_1)^2} \chi_{w_3,\ldots,w_{2\ell}}((x_3, x_4, \ldots, x_{2\ell})), \\ S_2 &= \sum_{H(x_3,\ldots,x_{2\ell}) = 1 + \alpha + \pi(\beta_1)^2} \chi_{w_3,\ldots,w_{2\ell}}((x_3, x_4, \ldots, x_{2\ell})). \end{aligned}$$

The sums S_1 and S_2 can be evaluated. If $\alpha+\pi(\beta_1)^2=0$, then in the sum for S_1, we are summing over all $(x_3,\ldots,x_{2\ell})$ satisfying $H(x_3,\ldots,x_{2\ell})=0$, including the element $(0,0,\ldots,0)$. So

$$S_1 = 1+\sum_{H(x_3,\ldots,x_{2\ell})=0,(x_3,\ldots,x_{2\ell})\neq(0,0,\ldots 0)} \chi_{w_3,\ldots,w_{2\ell}}((x_3,x_4,\ldots,x_{2\ell})).$$

Now note that the subset $\{(x_3,\ldots,x_{2\ell}) \mid H(x_3,\ldots,x_{2\ell})=0,(x_3,\ldots,x_{2\ell})\neq(0,0,\ldots 0)\}$ is a Latin square type PDS in $\mathbb{F}_4^{2\ell-2}$ (this follows from the hyperbolic version of Theorem 1.5). By Lemma 1.3, we have

$$S_1 = \begin{cases} 4^{2\ell-3}+4^{\ell-1}-4^{\ell-2}, & \text{if } (w_3,w_4,\ldots,w_{2\ell})=(0,0,\ldots,0),\\ 4^{\ell-1}-4^{\ell-2} \text{ or } -4^{\ell-2}, & \text{otherwise.}\end{cases} \tag{3.9}$$

If $\alpha+\pi(\beta_1)^2\neq 0$, then by Theorem 1.6 (with the quadratic form Q in that theorem being H), we have

$$S_1 = \begin{cases} 4^{2\ell-3}-4^{\ell-2}, & \text{if } (w_3,w_4,\ldots,w_{2\ell})=(0,0,\ldots,0),\\ 4^{\ell-1}-4^{\ell-2} \text{ or } -4^{\ell-2}, & \text{otherwise.}\end{cases} \tag{3.10}$$

The same is true for S_2. That is, if $1+\alpha+\pi(\beta_1)^2=0$, then

$$S_2 = \begin{cases} 4^{2\ell-3}+4^{\ell-1}-4^{\ell-2}, & \text{if } (w_3,w_4,\ldots,w_{2\ell})=(0,0,\ldots,0),\\ 4^{\ell-1}-4^{\ell-2} \text{ or } -4^{\ell-2}, & \text{otherwise.}\end{cases} \tag{3.11}$$

And if $1+\alpha+\pi(\beta_1)^2\neq 0$,

$$S_2 = \begin{cases} 4^{2\ell-3}-4^{\ell-2}, & \text{if } (w_3,w_4,\ldots,w_{2\ell})=(0,0,\ldots,0),\\ 4^{\ell-1}-4^{\ell-2} \text{ or } -4^{\ell-2}, & \text{otherwise.}\end{cases} \tag{3.12}$$

If $(w_3,w_4,\ldots,w_{2\ell})=(0,0,\ldots,0)$, we contend that $S_1=S_2=4^{2\ell-3}-4^{\ell-2}$. The reason is that all other choices for S_1 and S_2 result in

$$S_1+S_2=(4^{2\ell-3}+4^{\ell-1}-4^{\ell-2})+(4^{2\ell-3}-4^{\ell-2})=2(4^{2\ell-3}-4^{\ell-2})+4^{\ell-1},$$

which is never equal to $(4^{2\ell-2}-4^{\ell-1})/2$ as required by (3.8). Hence in this case $\chi_{\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell}}(D_1\backslash O_0)=0$ by (3.5).

If $(w_3,w_4,\ldots,w_{2\ell})\neq(0,0,\ldots,0)$, we contend that $S_1=S_2=-4^{\ell-2}$. The reason is that all other choices for S_1 and S_2 result in either

$$S_1+S_2=4^{\ell-1}-4^{\ell-2}+(-4^{\ell-2})=2\cdot 4^{\ell-2},$$

or

$$S_1+S_2=2(4^{\ell-1}-4^{\ell-2})=6\cdot 4^{\ell-2}.$$

Neither of these two values equals $-2\cdot 4^{\ell-2}$ as required by (3.8). Hence in this case we also have $\chi_{\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell}}(D_1\backslash O_0)=0$ by (3.5). This completes the proof of the lemma. □

Corollary 3.4. *If $\chi_{\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell}}(O_0)=4^{\ell-1}$, then $\Psi(\mathcal{L}_1)=4^{\ell-1}$.*

Proof: Use (3.2) and Lemma 3.3. □

Combining the discussion on characters of order 2, Corollary 3.2 and Corollary 3.4, we have the following theorem.

Theorem 3.5. *The set $\mathcal{L}_1$ is a negative Latin square type PDS in the additive group of $R\times\mathbb{F}_4^{2\ell-2}$.*

Proof: Let Ψ be a character of the additive group of $R \times \mathbb{F}_4^{2\ell-2}$. As we see in the above, the character sum $\Psi(\mathcal{L}_1)$ has the correct values in all cases. The theorem follows from Lemma 1.3. $\square$

As in Section 2, the arguments to show that $\mathcal{L}_\alpha$ and $\mathcal{L}_{\alpha^2}$ are negative Latin square type PDSs will go through in analogous ways by considering the quadratic forms αQ and $\alpha^2 Q$. The subset $\mathcal{L}_0 \setminus \{0\}$ was already shown in [7] to be a negative Latin square type PDS. This leads to the main result of this section.

Theorem 3.6. *The strongly regular Cayley graphs associated to the negative Latin square type PDSs $\mathcal{L}_0 \setminus \{0\}, \mathcal{L}_1, \mathcal{L}_\alpha,$ and $\mathcal{L}_{\alpha^2}$ form a 4-class amorphic association scheme on $R \times \mathbb{F}_4^{2\ell-2}$.*

Proof: Note that $\mathcal{L}_0 \setminus \{0\}, \mathcal{L}_1, \mathcal{L}_\alpha,$ and $\mathcal{L}_{\alpha^2}$ partition $R \times \mathbb{F}_4^{2\ell-2} \setminus \{0\}$. Each of them is a negative Latin square type PDS. By Theorem 1.2, the strongly regular Cayley graphs associated with these four subsets form an amorphic association scheme on $R \times \mathbb{F}_4^{2\ell-2}$. $\square$

As indicated in the introduction, this association scheme seems to be the first amorphic association scheme with negative Latin square type graphs defined on a nonelementary abelian group. (We remark that there are known constructions of amorphic association schemes with Latin square type graphs defined on the additive groups of Galois rings. See [10].) We could extend this result to more copies of R using techniques similar to that in [7], but the computations are tedious. We therefore will not include the details here. The interested reader is referred to [7] for details on how to construct PDSs in $R^k \times \mathbb{F}_4^{2\ell-2k}$, where $1 < k < \ell$.

4. Further amorphic schemes from Brouwer's construction

Let q be a prime power, and let m, ℓ be positive integers. Suppose that m has a chain of divisors

$$1 = m_d | m_{d-1} | m_{d-2} | \cdots | m_2 | m_1 = m,$$

where $m_i \neq m_{i-1}$ for all $1 < i \leq d$. Then the finite field $\mathbb{F}_{q^m}$ has a chain of subfields

$$\mathbb{F}_q = \mathbb{F}_{q^{m_d}} \subset \mathbb{F}_{q^{m_{d-1}}} \subset \cdots \subset \mathbb{F}_{q^{m_2}} \subset \mathbb{F}_{q^{m_1}} = \mathbb{F}_{q^m}.$$

Let $Q : V \to \mathbb{F}_{q^m}$ be a nonsingular elliptic quadratic form, where $V = \mathbb{F}_{q^m}^{2\ell}$, and let $\mathrm{tr}_{q^m/q^{m_i}}$ be the trace from $\mathbb{F}_{q^m}$ to $\mathbb{F}_{q^{m_i}}$. For $1 \leq i \leq d$, we define

$$Q_i = \mathrm{tr}_{q^m/q^{m_i}} \circ Q,$$

and

$$\Omega_i = \{x \in V \mid x \neq 0, Q_i(x) = 0\}.$$

Then each $Q_i : V \to \mathbb{F}_{q^{m_i}}$ is a nonsingular elliptic quadratic form (cf. [3]), and by transitivity of the traces, we have

$$\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_d.$$

By Theorem 1.5 each Ω_i is a negative Latin square type PDS, and Theorem 1.6 shows that $\Omega_i \backslash \Omega_j$ is a negative Latin square type PDS for all $i > j$. We can prove the following theorem:

Theorem 4.1. *Let $Q : V \to \mathbb{F}_{q^m}$ be a nonsingular elliptic quadratic form, where $V = \mathbb{F}_{q^m}^{2\ell}$, and let $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_d$ be the sets described above. Then the Cayley graphs generated by the collection $\Omega_1, \Omega_2 \backslash \Omega_1, \Omega_3 \backslash \Omega_2, \ldots, \Omega_d \backslash \Omega_{d-1}, (V \setminus \{(0, 0, \ldots, 0)\}) \backslash \Omega_d$ form a $(d+1)$-class amorphic association scheme on V.*

Proof: Theorem 1.6 implies that $\Omega_i \backslash \Omega_{i-1}$, $1 < i \leq d$, are negative Latin square type PDSs when Q is elliptic. In addition, the complement of Ω_d in $V \setminus \{(0,0,\ldots,0)\}$ is also a PDS of the same type. Thus, we have partitioned $V \setminus \{(0,0,\ldots,0)\}$ into disjoint PDSs with negative Latin square type parameters. The strongly regular Cayley graphs associated with $\Omega_1, \Omega_2\backslash\Omega_1, \Omega_3\backslash\Omega_2, \ldots, \Omega_d\backslash\Omega_{d-1}, (V \setminus \{(0,0,\ldots,0)\})\backslash\Omega_d$ are all of negative Latin square type, and form an edge disjoint decomposition of the complete graph on the vertex set V. Theorem 1.2 implies that these must be the classes of an amorphic association scheme. □

This theorem also works for hyperbolic quadratic forms, producing Latin square type strongly regular graphs as the classes of the amorphic association scheme. We note that this theorem contains some ambiguity regarding the choice of the subfield chain. For example, if we are working with the field $\mathbb{F}_{p^{30}}$, where p is a prime, we could use any of the following chains: $\mathbb{F}_p \subset \mathbb{F}_{p^3} \subset \mathbb{F}_{p^6} \subset \mathbb{F}_{p^{30}}; \mathbb{F}_p \subset \mathbb{F}_{p^3} \subset \mathbb{F}_{p^{15}} \subset \mathbb{F}_{p^{30}}; \mathbb{F}_p \subset \mathbb{F}_{p^5} \subset \mathbb{F}_{p^{10}} \subset \mathbb{F}_{p^{30}}; \mathbb{F}_p \subset \mathbb{F}_{p^5} \subset \mathbb{F}_{p^{15}} \subset \mathbb{F}_{p^{30}}; \mathbb{F}_p \subset \mathbb{F}_{p^2} \subset \mathbb{F}_{p^6} \subset \mathbb{F}_{p^{30}}; \mathbb{F}_p \subset \mathbb{F}_{p^2} \subset \mathbb{F}_{p^{10}} \subset \mathbb{F}_{p^{30}}$. Theorem 4.1 applies to each of these chains to produce a 5-class amorphic association scheme, but the actual strongly regular graphs will be different for each choice.

Theorem 4.1 provides an alternative proof of the following result originally found in Hamilton's paper [9].

Corollary 4.2. *Let $Q : V \to \mathbb{F}_{q^m}$ be a nonsingular elliptic quadratic form, where $V = \mathbb{F}_{q^m}^{2\ell}$, and let $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_d$ be described as above. Then $(\Omega_d\backslash\Omega_{d-1}) \cup (\Omega_{d-2}\backslash\Omega_{d-3}) \cup \cdots \cup (\Omega_2\backslash\Omega_1)$ is a negative Latin square type PDS if d is even and $(\Omega_d\backslash\Omega_{d-1}) \cup (\Omega_{d-2}\backslash\Omega_{d-3}) \cup \cdots \cup (\Omega_3\backslash\Omega_2) \cup \Omega_1$ is a negative Latin square type PDS if d is odd.*

Proof: Simply fuse the appropriate classes from the amorphic association scheme in Theorem 4.1. □

The simple proof of Corollary 4.2 given above demonstrates the power of Theorem 1.2. We show one more application of this theorem by doing a variation of Theorem 4.1 for quadratic forms with values in $\mathbb{F}_{q^m}$, where m is even. Without loss of generality, we will assume $m = 2$. The result will be a generalization of Theorem 2.1 in Section 2. Before we state the result, we need the following lemma identifying the trace 0 elements in a quadratic extension field of $\mathbb{F}_q$.

Lemma 4.3.
1. *Let $s \geq 1$ be an integer, $q = 2^s$, and let g be a primitive element of $\mathbb{F}_{q^2}$. Then $\{x \in \mathbb{F}_{q^2} \mid \mathrm{tr}_{q^2/q}(x) = 0\} = \{0\} \cup \{g^{(q+1)i} \mid 0 \leq i < q-1\}$.*
2. *Let $s \geq 1$ be an integer, p an odd prime, $q = p^s$, and let g be a primitive element of $\mathbb{F}_{q^2}$. Then $\{x \in \mathbb{F}_{q^2} \mid \mathrm{tr}_{q^2/q}(x) = 0\} = \{0\} \cup \{g^{\frac{q+1}{2}+(q+1)i} \mid 0 \leq i < q-1\}$.*

Proof: In both cases, we have $\mathrm{tr}_{q^2/q}(x) = x + x^q$ for all $x \in \mathbb{F}_{q^2}$. In the $q = 2^s$ case, note that $g^{(q+1)i} \in \mathbb{F}_q \subset \mathbb{F}_{q^2}$, hence $\mathrm{tr}_{q^2/q}(g^{(q+1)i}) = g^{(q+1)i} + (g^{(q+1)i})^q = g^{(q+1)i} + g^{(q+1)i} = 0$. In the $q = p^s$ case, where p is an odd prime, again noting that $g^{(q+1)i} \in \mathbb{F}_q \subset \mathbb{F}_{q^2}$, we have

$$\mathrm{tr}_{q^2/q}(g^{\frac{q+1}{2}+(q+1)i}) = g^{(q+1)i}(g^{\frac{q+1}{2}} + g^{\frac{(q+1)q}{2}}) = g^{(q+1)i}(g^{\frac{q+1}{2}} + g^{\frac{q^2-1}{2}} g^{\frac{q+1}{2}}) = 0,$$

since $g^{\frac{q^2-1}{2}} = -1$. A counting argument indicates that we have found all of the trace 0 elements. □

We now vary the quadratic form in a way similar to our approach in Section 2: if $Q : \mathbb{F}_{q^2}^{2\ell} \to \mathbb{F}_{q^2}$ is a nonsingular elliptic quadratic form and g is a primitive element of $\mathbb{F}_{q^2}$, then consider the $q+1$ quadratic forms $\mathrm{tr}_{q^2/q} \circ g^i Q, 0 \leq i \leq q$. Let

$$\Omega_{g^i} = \{x \in \mathbb{F}_{q^2}^{2\ell} \mid \mathrm{tr}_{q^2/q} \circ g^i Q(x) = 0, x \neq 0\},$$

for all $0 \leq i \leq q$. Each of these sets contains the same subset $\Omega_0 = \{x \in \mathbb{F}_{q^2}^{2\ell} \mid x \neq 0, Q(x) = 0\}$ associated to the quadratic form Q since $Q(x) = 0$ if and only if $g^i Q(x) = 0$. Theorem 1.6 implies that $\Omega_{g^i} \setminus \Omega_0$ is a negative Latin square type PDS for all $i = 0, 1, \ldots, q$. This leads to the following theorem.

Theorem 4.4. *Let g be a primitive element of $\mathbb{F}_{q^2}$, and let $Q : \mathbb{F}_{q^2}^{2\ell} \to \mathbb{F}_{q^2}$ be a nonsingular elliptic quadratic form. Then the strongly regular graphs associated to the PDSs $\Omega_0, \Omega_{g^i} \setminus \Omega_0, 0 \leq i \leq q$, form a $(q+2)$-class amorphic association scheme.*

Proof: Each of the PDSs $\Omega_0, \Omega_{g^i} \setminus \Omega_0, 0 \leq i \leq q$, is of negative Latin square type, and hence the associated strongly regular Cayley graph is also of negative Latin square type. Lemma 4.3 implies that these negative Latin square type PDSs partition $\mathbb{F}_{q^2}^{2\ell} \setminus \{0\}$. That means the associated SRGs form an edge-decomposition of the complete graph on $\mathbb{F}_{q^2}^{2\ell}$. Now the result follows from Theorem 1.2. □

As an example, consider a nonsingular elliptic quadratic form with its values in $\mathbb{F}_{p^{30}}$. If we apply Theorem 4.1, we will get several 5-class amorphic association schemes. If we apply Theorem 4.4 with $q = p^{15}$, we get a $(p^{15} + 2)$-class amorphic association scheme. These can be fused to yield any of the 5-class amorphic association schemes from the previous construction.

Setting $q = 2$ in Theorem 4.4, we obtain Theorem 2.1. In Section 3, we "lifted" the PDSs $D_0 \setminus \{0\}$, D_1, D_α, and D_{α^2} in Theorem 2.1 to PDSs $\mathcal{L}_0 \setminus \{0\}$, $\mathcal{L}_1$, $\mathcal{L}_\alpha$, and $\mathcal{L}_{\alpha^2}$ in $(R \times \mathbb{F}_4^{2\ell-2}, +)$ that partition $(R \times \mathbb{F}_4^{2\ell-2}) \setminus \{(0, 0, \ldots, 0)\}$, hence obtained a 4-class amorphic association scheme defined on $(R \times \mathbb{F}_4^{2\ell-2}, +)$. It is natural to ask whether one can do the same for the PDSs in Theorem 4.4 when $q > 2$. Computer computations indicate that when $q > 2$ and $q = p^s$, p a prime, if we follow the same procedure to "lift" Ω_0 to $R' \times \mathbb{F}_{q^2}^{2\ell-2}$, where R' is the Galois ring $\mathrm{GR}(p^2, 2s)$, the resulting set is not a PDS.

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